Eur. Phys. J. C 51, 229–240 (2007)
DOI 10.1140/epjc/s10052-007-0285-8

THE EUROPEAN PHYSICAL JOURNAL C

Regular Article – Theoretical Physics

Covariant realizations of kappa-deformed space

S. Meljanac[1,a], S. Krešić-Jurić[2,b], M. Stojić[1,c]

[1] Rudjer Bošković Institute, Bijenička c. 54, 10002 Zagreb, Croatia
[2] Faculty of Natural and Mathematical Sciences, Teslina 12, 21000 Split, Croatia

Received: 26 February 2007 /
Published online: 18 April 2007 –

Abstract. We study a Lie algebra type κ-deformed space with an undeformed rotation algebra and commutative vector-like Dirac derivatives in a covariant way. The space deformation depends on an arbitrary vector. Infinitely many covariant realizations in terms of commuting coordinates of undeformed space and their derivatives are constructed. The corresponding coproducts and star products are found and related in a new way. All covariant realizations are physically equivalent. Specially, a few simple realizations are found and discussed. The scalar fields, invariants and the notion of invariant integration is discussed in the natural realization.

1 Introduction

In the previous decade there has been growing interest in the formulation of physical theories defined on noncommutative (NC) spaces. The consistency of such theories and their implications were studied in [1–9]. It is important to classify the NC spaces and investigate their properties, and, particularly, to develop a unifying approach to a generalized theory for such spaces that is convenient for physical applications. The notion of generalized symmetries and their role in the analysis of NC spaces is also crucial. In order to make a contribution in this direction, we analyze a NC space of the Lie algebra type, in particular the so-called κ-deformed space introduced in [10–12].

For simplicity, we restrict our attention to κ-deformed Euclidean space, although the analysis can easily be extended to κ-deformed Minkowski space. The noncommutativity of the coordinates depends on a deformation parameter, which is an arbitrary vector $a \in \mathbb{R}^n$. The dimensional parameter $|a| = 1/\kappa$ has a very small length, which yields the undeformed Euclidean space in the limit $|a| \to 0$. The NC coordinates and the generators of the generalized rotations form an extended Lie algebra. The subalgebra formed by the rotation generators is the ordinary $\mathrm{SO}_a(n)$ algebra, i.e. the ordinary Lorentz algebra in the case of κ-deformed Minkowski space. Dirac derivatives are assumed to commute mutually and transform as a vector representation under the $\mathrm{SO}_a(n)$ algebra. This κ-deformed space was studied by different groups, from both the mathematical and physical point of view [13–35]. There is also an interesting connection to the doubly special relativity program [19–22]. Realizations of NC spaces in terms of commutative coordinates and derivative operators have been obtained and discussed in the cases of symmetric ordering and normal (left and right) ordering of NC coordinates [17, 27]. An infinite family of noncovariant realizations was found in [28]. Although a single space may be realized in many different ways, physical results do not depend on concrete realizations, i.e. on the orderings [31].

Our aim in this paper is to construct covariant realizations for general κ-deformed Euclidean spaces depending on an arbitrary deformation vector a. We analyze such spaces by using the methods developed for deformed single and multimode oscillators in the Fock space representations [37–51]. In particular, we use the methods for constructing deformed creation and annihilation operators in terms of ordinary bosonic multimode oscillators, i.e. a kind of bosonization [37–39, 48]. Also, we employ the construction of transition number operators and, in general, the construction of generators as proposed in [39–42, 47].

The simple connection between creation and annihilation operators with NC coordinates and Dirac derivatives is established by using a Bargman type representation. We find infinitely many new covariant realizations in terms of commutative coordinates and derivative operators. The realizations depend on certain parameter functions, but they can all be treated on an equal footing, and the physical results do not depend on them. For a special choice of the parameter functions we obtain some particularly simple realizations: covariant left, right and natural realizations. These realizations are considered in detail, and a coproduct and star product are associated to each of them.

The outline of the paper is as follows. In Sect. 2 we introduce a Lie algebra type of κ-deformed Euclidean space. We also define the rotation algebra $\mathrm{SO}_a(n)$ that is compatible with κ-deformations, and we introduce the Dirac

[a] e-mail: meljanac@thphys.irb.hr, meljanac@irb.hr
[b] e-mail: skresic@fesb.hr
[c] e-mail: marko.stojic@zg.htnet.hr

derivative and the Laplace operator. The shift operator, which plays an important role in describing the algebra generated by NC coordinates, the rotation generators and the Dirac derivative are also introduced. Special consideration is given to the generalized Leibniz rule and coproduct. We derive expressions for the coproduct of the rotation generators and the Dirac derivative, and of the left and right deformations of the ordinary derivative. Section 3 deals with covariant realizations of the κ-deformed Euclidean space and the operators introduced in Sect. 2. We find an infinite family of covariant realizations in terms of commutative coordinates and derivative operators. In particular, we obtain two types of realizations (type I and type II), which depend on an arbitrary parameter function φ. For special choices of φ, we construct particularly simple covariant realizations: left, right, symmetric and natural realizations. In Sect. 4 we consider the star product for the realizations discussed in Sect. 3. A general formula for the star product is given to second order in the deformation parameter a, and closed form expressions are obtained in the left, right and symmetric realizations. We also introduce the notion of equivalent star products, using similarity transformations. We show that any star product in the realization of type I can be obtained from the star product in the right realization using similarity transformations. In Sect. 4.4 we introduce scalar fields in NC coordinates and demonstrate their simple properties in terms of the natural realization. Constructions of invariants and the invariant integration on NC spaces are also presented. Finally, in Sect. 5 a brief conclusion is given.

2 Kappa-deformed Euclidean space

Consider a Lie algebra type noncommutative (NC) space generated by coordinates $\hat{x}_1, \hat{x}_2, \dots, \hat{x}_n$ satisfying the commutation relations

$$[\hat{x}_\mu, \hat{x}_\nu] = \mathrm{i}(a_\mu \hat{x}_\nu - a_\nu \hat{x}_\mu)\,, \tag{1}$$

where $\mu, \nu = 1, \dots, n$ and $a_1, a_2, \dots, a_n$ are the components of a vector $a \in \mathbb{R}^n$ that describes a deformation of the Euclidean space [25–28]. The structure constants are given by

$$C_{\mu\nu\lambda} = a_\mu \delta_{\nu\lambda} - a_\nu \delta_{\mu\lambda}\,. \tag{2}$$

In the limit $a \to 0$, we have $\hat{x}_\mu \to x_\mu$, the ordinary commutative coordinates.

Let $\mathrm{SO}_a(n)$ be the ordinary rotation algebra with generators $M_{\mu\nu}$ satisfying

$$M_{\mu\nu} = -M_{\nu\mu}\,, \tag{3}$$

$$[M_{\mu\nu}, M_{\lambda\rho}] = \delta_{\nu\lambda} M_{\mu\rho} - \delta_{\mu\lambda} M_{\nu\rho} - \delta_{\nu\rho} M_{\mu\lambda} + \delta_{\mu\rho} M_{\nu\lambda}\,. \tag{4}$$

We require that the rotation generators $M_{\mu\nu}$ and the coordinates $\hat{x}_\mu$ form an extended Lie algebra. The most general form of the commutator $[M_{\mu\nu}, \hat{x}_\lambda]$ must be linear in $M_{\mu\nu}$ and $\hat{x}_\lambda$, antisymmetric in the indices μ and ν, and it must have the smooth limit $[M_{\mu\nu}, \hat{x}_\lambda] \to x_\mu \delta_{\nu\lambda} - x_\nu \delta_{\mu\lambda}$ as $a \to 0$. The required form is given by

$$\begin{aligned}[M_{\mu\nu}, \hat{x}_\lambda] = {} & \hat{x}_\mu \delta_{\nu\lambda} - \hat{x}_\nu \delta_{\mu\lambda} \\ & + \mathrm{i} s a_\lambda M_{\mu\nu} - \mathrm{i} t \left(a_\mu M_{\nu\lambda} - a_\nu M_{\mu\lambda}\right) \\ & + \mathrm{i} u a_\alpha \left(M_{\alpha\mu} \delta_{\nu\lambda} - M_{\alpha\nu} \delta_{\mu\lambda}\right)\end{aligned} \tag{5}$$

for some $s, t, u \in \mathbb{R}$, where summation over repeated indices is assumed. The necessary and sufficient condition for consistency of the extended Lie algebra is that the Jacobi identity holds for all combinations of the generators $M_{\mu\nu}$ and $\hat{x}_\lambda$. One can verify that this is satisfied for the unique values of the parameters $s = u = 0$ and $t = 1$ [28]. Hence,

$$[M_{\mu\nu}, \hat{x}_\lambda] = \hat{x}_\mu \delta_{\nu\lambda} - \hat{x}_\nu \delta_{\mu\lambda} - \mathrm{i}\left(a_\mu M_{\nu\lambda} - a_\nu M_{\mu\lambda}\right)\,. \tag{6}$$

Having introduced the rotation algebra $\mathrm{SO}_a(n)$, it is natural to consider the Dirac derivatives D_μ, satisfying

$$[D_\mu, D_\nu] = 0, \tag{7}$$

$$[M_{\mu\nu}, D_\lambda] = \delta_{\nu\lambda} D_\mu - \delta_{\mu\lambda} D_\nu\,. \tag{8}$$

The generators M_μ and D_λ form the undeformed $\mathrm{ISO}_a(n)$ algebra, i.e. the Poincaré algebra in the case of κ-deformed Minkowski space. Note that the operator $D^2 = D_\mu D_\mu$ is invariant under rotations since $[M_{\mu\nu}, D^2] = 0$.

We also wish to define commutation relations for D_μ and $\hat{x}_\nu$. The consistency condition is that the Jacobi identity is satisfied for all combinations of the generators $M_{\mu\nu}$, D_λ and $\hat{x}_\rho$. It can be shown [28] that the correct form of the commutator is given by

$$[D_\mu, \hat{x}_\nu] = \delta_{\mu\nu} \sqrt{1 - a^2 D^2} + \mathrm{i} C_{\mu\alpha\nu} D_\alpha\,. \tag{9}$$

The algebra generated by D_μ and $\hat{x}_\nu$ is a deformed Heisenberg algebra. We note that, in the limit $a \to 0$, the commutation relations (1), (7) and (9) yield the ordinary undeformed Heisenberg algebra. Hence, $D_\mu \to \partial_\mu$ and $\hat{x}_\mu \to x_\mu$ as $a \to 0$, where $\partial_\mu = \frac{\partial}{\partial x_\mu}$. Particularly, in the one-dimensional case, $n = 1$, (9) leads to a generalized uncertainty relation with minimal length [52–54].

A function $f(\hat{x}, D)$, where $f(\hat{x}, D)$ denotes a formal power series in the monomials $\hat{x}_{\mu_1} \hat{x}_{\mu_2} \dots \hat{x}_{\mu_n}$ and $D_{\nu_1} D_{\nu_2} \dots D_{\nu_m}$, is said to be $\mathrm{SO}_a(n)$ invariant if

$$[M_{\mu\nu}, f(\hat{x}, D)] = 0 \quad \text{for all} \quad \mu, \nu\,. \tag{10}$$

We introduce the $\mathrm{SO}_a(n)$ invariant Laplace operator $\Box$ satisfying the commutation relations

$$[M_{\mu\nu}, \Box] = 0\,, \tag{11}$$

$$[\Box, \hat{x}_\mu] = 2 D_\mu\,. \tag{12}$$

The Laplace operator can be expressed in terms of the operator D^2. Let us assume the Ansatz $\Box = F(D^2)$, where

F is analytic, and impose the boundary condition of $F(D^2) \to \partial^2$ as $a \to 0$. Then (11) is automatically satisfied, and (9) and (12) together with the boundary condition yield

$$\Box = \frac{2}{a^2}\left(1-\sqrt{1-a^2D^2}\right) . \tag{13}$$

It follows that

$$D^2 = \Box\left(1-\frac{a^2}{4}\Box\right) . \tag{14}$$

2.1 The shift operator

At this point it is convenient to introduce the shift operator Z via the commutation relations

$$[Z, \hat{x}_\mu] = \mathrm{i}a_\mu Z \,, \tag{15}$$

$$[Z, D_\mu] = 0 \,. \tag{16}$$

We assume that $Z \to 1$ in the limit $a \to 0$. The shift operator acts on an arbitrary function $f(\hat{x})$ by

$$Zf(\hat{x}) = f(\hat{x}+\mathrm{i}a)Z \,. \tag{17}$$

We note that the inverse shift operator Z^{-1} satisfies the same relations as Z if a_μ is replaced by $-a_\mu$. The shift operator can be expressed in terms of the Dirac derivative D as follows. Using (9) and (12) one can verify that

$$\left[-\mathrm{i}aD - \frac{a^2}{2}\Box, \hat{x}_\mu\right] = -\mathrm{i}a_\mu\left(-\mathrm{i}aD + \sqrt{1-a^2D^2}\right) , \tag{18}$$

where $aD = a_\mu D_\mu$. Inserting (13) for the Laplace operator into (18), we obtain $[Z^{-1}, \hat{x}_\mu] = -\mathrm{i}a_\mu Z^{-1}$, where

$$Z^{-1} = -\mathrm{i}aD + \sqrt{1-a^2D^2} \,. \tag{19}$$

Inverting the above expression, we find

$$Z = \frac{\mathrm{i}aD + \sqrt{1-a^2D^2}}{1-a^2D^2+(aD)^2} \,. \tag{20}$$

It is interesting to note that the algebra generated by $\hat{x}_\mu$, $M_{\mu\nu}$ and D_μ can be described using the shift operator Z by the relations

$$\hat{x}_\mu Z \hat{x}_\nu = x_\nu Z \hat{x}_\mu \,, \tag{21}$$

$$[D_\mu, \hat{x}_\nu] = \delta_{\mu\nu} Z^{-1} + \mathrm{i}a_\mu D_\nu \,. \tag{22}$$

The remaining commutation relations for $[M_{\mu\nu}, M_{\lambda\rho}]$, $[M_{\mu\nu}, \hat{x}_\lambda]$ and $[M_{\mu\nu}, D_\lambda]$ are satisfied by representing $M_{\mu\nu}$ in the unique way as

$$M_{\mu\nu} = (\hat{x}_\mu D_\nu - \hat{x}_\nu D_\mu)Z \,. \tag{23}$$

We will justify this relation later when we consider the so-called natural realization in Sect. 3.

2.2 The Leibniz rule and coproduct

Now we turn our attention to the generalized Leibniz rule and coproduct. The commutator of $M_{\mu\nu}$ with an arbitrary function $f(\hat{x})$ is given by

$$\begin{aligned}[M_{\mu\nu}, f] = (M_{\mu\nu}f) + \mathrm{i}a_\mu \left[\left(D_\lambda - \frac{\mathrm{i}a_\lambda}{2}\Box\right) Zf\right] M_{\lambda\nu} \\ - \mathrm{i}a_\nu \left[\left(D_\lambda - \frac{\mathrm{i}a_\lambda}{2}\Box\right) Zf\right] M_{\lambda\mu} \,. \end{aligned} \tag{24}$$

This relation can be shown by using (6) and proceeding by induction on the degree of the monomials in $\hat{x}_\mu$. From (24) we obtain the coproduct for $M_{\mu\nu}$,

$$\begin{aligned}\triangle M_{\mu\nu} = M_{\mu\nu} \otimes \mathbf{1} + \mathbf{1} \otimes M_{\mu\nu} \\ + \mathrm{i}a_\mu \left(D_\lambda - \frac{\mathrm{i}a_\lambda}{2}\Box\right) Z \otimes M_{\lambda\nu} \\ - \mathrm{i}a_\nu \left(D_\lambda - \frac{\mathrm{i}a_\lambda}{2}\Box\right) Z \otimes M_{\lambda\mu} \,. \end{aligned} \tag{25}$$

Similarly, one can show that the commutator of D_μ with $f(\hat{x})$ is given by

$$[D_\mu, f] = (D_\mu f)Z^{-1} + \mathrm{i}a_\mu(D_\lambda Zf)D_\lambda - \frac{\mathrm{i}a_\mu}{2}(\Box Zf)\mathrm{i}aD \,; \tag{26}$$

hence we have for the coproduct

$$\begin{aligned}\triangle D_\mu = D_\mu \otimes Z^{-1} + \mathbf{1} \otimes D_\mu + \mathrm{i}a_\mu(D_\lambda Z) \otimes D_\lambda \\ - \frac{\mathrm{i}a_\mu}{2}\Box Z \otimes \mathrm{i}aD \,. \end{aligned} \tag{27}$$

Furthermore, the coproduct for the shift operator Z is simply

$$\triangle Z = Z \otimes Z \,. \tag{28}$$

Some examples of the Poincaré invariant interpretation of NC spaces and of the twisted Poincaré coalgebra were also considered in [55–58].

It is interesting to note that the operators D_μ, $\Box$ and Z can be expressed in terms of the auxiliary derivatives $\partial^{\mathrm{L}}_\mu$ and $\partial^{\mathrm{R}}_\mu$ satisfying the following commutation relations:

$$\left[\partial^{\mathrm{L}}_\mu, \partial^{\mathrm{L}}_\nu\right] = 0 \,, \tag{29}$$

$$\left[\partial^{\mathrm{L}}_\mu, \hat{x}_\nu\right] = \delta_{\mu\nu} Z^{-1} \tag{30}$$

and

$$\left[\partial^{\mathrm{R}}_\mu, \partial^{\mathrm{R}}_\nu\right] = 0 \,, \tag{31}$$

$$\left[\partial^{\mathrm{R}}_\mu, \hat{x}_\nu\right] = \delta_{\mu\nu} + \mathrm{i}a_\nu \partial^{\mathrm{R}}_\mu \,. \tag{32}$$

Equations (29)–(30) and (31)–(32) are consistent with the commutation relation (1), and hence both sets of equations define a deformed Heisenberg algebra. One can think of $\partial^{\mathrm{L}}_\mu$ and $\partial^{\mathrm{R}}_\mu$ as "left" and "right" deformations of the ordinary derivative ∂_μ. Indeed, using the commutation relations (30) and (32), one can show that the coproducts of $\partial_\mu^{\mathrm{L}}$ and $\partial_\mu^{\mathrm{R}}$ are given by

$$\triangle\partial_\mu^{\mathrm{L}} = \partial_\mu^{\mathrm{L}} \otimes Z^{-1} + \mathbf{1} \otimes \partial_\mu^{\mathrm{L}}, \tag{33}$$
$$\triangle\partial_\mu^{\mathrm{R}} = \partial_\mu^{\mathrm{R}} \otimes \mathbf{1} + Z \otimes \partial_\mu^{\mathrm{R}} \tag{34}$$

and $\partial_\mu^{\mathrm{R}} = \partial_\mu^{\mathrm{L}} Z$. Hence, the coproducts $\triangle\partial_\mu^{\mathrm{L}}$ and $\triangle\partial_\mu^{\mathrm{R}}$ are left and right deformations of the ordinary coproduct $\triangle\partial_\mu = \partial_\mu \otimes \mathbf{1} + \mathbf{1} \otimes \partial_\mu$, respectively. In fact, the coproducts $\triangle\partial_\mu^{\mathrm{L}}$ and $\triangle\partial_\mu^{\mathrm{R}}$ are given by (33)–(34) if and only if (30) and (32) hold. One can show that the operators D_μ, $\Box$ and Z are expressed in terms of the left and right deformation derivatives as

$$D_\mu = \partial_\mu^{\mathrm{L}} + \frac{\mathrm{i}a_\mu}{2}\Box\,, \tag{35}$$
$$\Box = (\partial^{\mathrm{L}})^2 Z\,, \tag{36}$$
$$Z = 1 + \mathrm{i}(a\partial^{\mathrm{L}})Z = \frac{1}{1 - \mathrm{i}a\partial^{\mathrm{L}}} \tag{37}$$

and

$$D_\mu = \partial_\mu^{\mathrm{R}} Z^{-1} + \frac{\mathrm{i}a_\mu}{2}\Box\,, \tag{38}$$
$$\Box = (\partial^{\mathrm{R}})^2 Z^{-1}\,, \tag{39}$$
$$Z = 1 + \mathrm{i}a\partial^{\mathrm{R}}\,. \tag{40}$$

The algebra generated by $\hat{x}_\mu$, $M_{\mu\nu}$ and D_μ is covariant under the action of the rotation group $\mathrm{SO}(n)$. Indeed, let $R \in \mathrm{SO}(n)$ be a rotation matrix, and let us denote the transformed variables by $\hat{x}'_\mu = R_{\mu\alpha}\hat{x}_\alpha$, $D'_\mu = R_{\mu\alpha}D_\alpha$, $M'_{\mu\nu} = R_{\mu\alpha}R_{\nu\beta}M_{\alpha\beta}$ and $a'_\mu = R_{\mu\alpha}a_\alpha$. Then (13) and (20) immediately yield

$$\Box' = \Box \quad \text{and} \quad Z' = Z\,, \tag{41}$$

and the transformed generators $\hat{x}'_\mu$, $M'_{\mu\nu}$ and D'_μ satisfy the relations

$$\hat{x}'_\mu Z \hat{x}'_\nu = \hat{x}'_\nu Z \hat{x}'_\mu\,, \tag{42}$$
$$\left[D'_\mu, \hat{x}'_\nu\right] = \delta_{\mu\nu} Z^{-1} + \mathrm{i}a'_\mu D'_\nu\,, \tag{43}$$
$$M'_{\mu\nu} = (\hat{x}'_\mu D'_\nu - \hat{x}'_\nu D'_\mu) Z\,. \tag{44}$$

3 Covariant realizations

A realization of the NC coordinates $\hat{x}_\mu$ in terms of ordinary commutative coordinates and their derivatives was found using the Bargman representation and the methods developed in [37–39, 47, 48]. The goal of this section is to find covariant realizations of the algebra generated by the NC coordinates $\hat{x}_\mu$, the rotation generators $M_{\mu\nu}$ and the Dirac derivatives D_μ. The realizations are found in terms of functions of the ordinary coordinates $x_1, x_2, \ldots, x_n$ and their derivatives $\partial_1, \partial_2, \ldots, \partial_n$, which generate the Heisenberg algebra $[x_\mu, x_\nu] = [\partial_\mu, \partial_\nu] = 0$ and $[\partial_\mu, x_\nu] = \delta_{\mu\nu}$. In general, these functions will satisfy a system of coupled partial differential equations (PDEs) determined by the commutation relations for $\hat{x}_\mu$, $M_{\mu\nu}$ and D_μ. In the following we derive such systems of PDEs, and next we consider their solutions.

The most general Ansatz for $\hat{x}_\mu$ is

$$\hat{x}_\mu = x_\alpha \Phi_{\alpha\mu}(A, B)\,, \tag{45}$$

where $\Phi_{\alpha\mu}$ is a function of the commuting variables $A = \mathrm{i}a\partial$ and $B = a^2\partial^2$, and it satisfies the boundary condition $\Phi_{\alpha\mu}(0,0) = \delta_{\alpha\mu}$. This realization is covariant under the orthogonal transformation $R \in \mathrm{SO}(n)$ (c.f. Sect. 2.2 and $x'_\mu = R_{\mu\alpha}x_\alpha$, $\partial'_\nu = R_{\mu\alpha}\partial_\alpha$), i.e. under the action of the generators,

$$M^0_{\mu\nu} = x_\mu\partial_\nu - x_\nu\partial_\mu + a_\mu\frac{\partial}{\partial a_\nu} - a_\nu\frac{\partial}{\partial a_\mu}\,. \tag{46}$$

We consider the particular form of the above Ansatz given by

$$\hat{x}_\mu = x_\mu\varphi + \mathrm{i}(ax)\left(\partial_\mu\beta_1 + \mathrm{i}a_\mu\partial^2\beta_2\right) + \mathrm{i}(x\partial)\left(a_\mu\gamma_1 + \mathrm{i}a^2\partial_\mu\gamma_2\right)\,, \tag{47}$$

where φ, β_i and γ_i are functions of A and B. We impose the boundary conditions of $\varphi(0,0) = 1$ and of $\beta_i(0,0)$ and $\gamma_i(0,0)$ being finite in order to ensure the smooth limit $\hat{x}_\mu \to x_\mu$ as $a \to 0$. Substituting the Ansatz (47) into (1), we obtain the following system of differential equations:

$$\frac{\partial\varphi}{\partial A}\varphi - B\left(\frac{\partial\varphi}{\partial A} - 2A\frac{\partial\varphi}{\partial B}\right)\beta_2 + \left(A\frac{\partial\varphi}{\partial A} + 2B\frac{\partial\varphi}{\partial B}\right)\gamma_1 - \varphi(\gamma_1 - 1) = 0\,, \tag{48}$$

$$2\frac{\partial\varphi}{\partial B}\varphi - \left(\frac{\partial\varphi}{\partial A} - 2A\frac{\partial\varphi}{\partial B}\right)\beta_1 - \left(A\frac{\partial\varphi}{\partial A} + 2B\frac{\partial\varphi}{\partial B}\right)\gamma_2 + \varphi\gamma_2 = 0\,, \tag{49}$$

$$\begin{aligned}&\left(\frac{\partial\beta_1}{\partial A} - 2B\frac{\partial\beta_2}{\partial B}\right)\varphi - B\left(\frac{\partial\beta_1}{\partial A} - 2A\frac{\partial\beta_1}{\partial B}\right)\beta_2 \\ &+ B\left(\frac{\partial\beta_2}{\partial A} - 2A\frac{\partial\beta_2}{\partial B}\right)\beta_1 + \left(A\frac{\partial\beta_1}{\partial A} + 2B\frac{\partial\beta_2}{\partial B}\right)\gamma_1 \\ &+ B\left(A\frac{\partial\beta_2}{\partial A} + 2B\frac{\partial\beta_2}{\partial B}\right)\gamma_2 \\ &- \left(\beta_1^2 + 2A\beta_1\beta_2\right) + B\beta_2\gamma_2 - 2\beta_2\varphi + \beta_1 = 0\,,\end{aligned} \tag{50}$$

$$\begin{aligned}&-\left(2\frac{\partial\gamma_1}{\partial B} + \frac{\partial\gamma_2}{\partial A}\right)\varphi + \left(\frac{\partial\gamma_1}{\partial A} - 2A\frac{\partial\gamma_1}{\partial B}\right)\beta_1 \\ &+ \left(\frac{\partial\gamma_2}{\partial A} - 2A\frac{\partial\gamma_2}{\partial B}\right)B\beta_2 + \left(A\frac{\partial\gamma_1}{\partial A} + 2B\frac{\partial\gamma_1}{\partial B}\right)\gamma_2 \\ &- \left(A\frac{\partial\gamma_2}{\partial A} + 2B\frac{\partial\gamma_2}{\partial B}\right)\gamma_1 + \gamma_2(\beta_1 - \gamma_1 - 1) = 0\,.\end{aligned} \tag{51}$$

Next, we consider realizations of the rotation algebra $\mathrm{SO}_a(n)$. We assume that the rotation generators are given by the Ansatz

$$M_{\mu\nu} = (x_\mu\partial_\nu - x_\nu\partial_\mu)\mathcal{F}_1 + \mathrm{i}(x\partial)(a_\mu\partial_\nu - a_\nu\partial_\mu)\mathcal{F}_2 + \mathrm{i}(x_\mu a_\nu - x_\nu a_\mu)\partial^2\mathcal{F}_3\,, \tag{52}$$

where the functions $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$ depend on A and B and satisfy the boundary conditions of $\mathcal{F}_1(0,0)=1$ and of $F_2(0,0)$ and $\mathcal{F}_3(0,0)$ being finite, respectively. The boundary conditions imply that $M_{\mu\nu}$ becomes the ordinary rotation generator as $a\to 0$. The Ansatz is antisymmetric in the indices μ and ν. Now we can calculate the commutator $[M_{\mu\nu}, M_{\lambda\rho}]$ by inserting the Ansatz (52) into (4). This results in the system of equations

$$\mathcal{F}_1 = 1, \quad \frac{\partial\mathcal{F}_3}{\partial A} + \left(\mathcal{F}_3 + A\frac{\partial\mathcal{F}_3}{\partial A}\right)\mathcal{F}_2 - 2\mathcal{F}_3^2 = 0\,. \tag{53}$$

Since $\mathcal{F}_2$ is uniquely determined by $\mathcal{F}_3$, these equations provide a realization of the algebra $\mathrm{SO}_a(n)$ in terms of an arbitrary parameter function $\mathcal{F}_3$. However, $M_{\mu\nu}$ and $\hat{x}_\mu$ are required to form the extended Lie algebra (6); hence $\mathcal{F}_3$ is related to the functions φ, β_i and γ_i. This relation can be shown either directly from (6) or by using a realization of the Dirac operator D_μ and (23).

We assume that the Dirac operator is given by

$$D_\mu = \partial_\mu\mathcal{G}_1 + \mathrm{i}a_\mu\partial^2\mathcal{G}_2\,. \tag{54}$$

Here the functions $\mathcal{G}_1$ and $\mathcal{G}_2$ depend on A and B and satisfy the boundary conditions of $\mathcal{G}_1(0,0)=1$ and of $\mathcal{G}_2(0,0)$ being finite, respectively. Inserting (54) into (9), we obtain

$$\sqrt{1-a^2D^2} - \mathrm{i}aD = \mathcal{G}_1\varphi\,, \tag{55}$$

$$\frac{\partial\mathcal{G}_1}{\partial A}\varphi + \left(-\frac{\partial\mathcal{G}_1}{\partial A} + 2A\frac{\partial\mathcal{G}_1}{\partial B}\right)B\beta_2 + \left(\mathcal{G}_1 + A\frac{\partial\mathcal{G}_1}{\partial A} + 2B\frac{\partial\mathcal{G}_1}{\partial B}\right)\gamma_1 = 0\,, \tag{56}$$

$$2\frac{\partial\mathcal{G}_1}{\partial B}\varphi + \left(2A\frac{\partial\mathcal{G}_1}{\partial B} - \frac{\partial\mathcal{G}_1}{\partial A}\right)\beta_1 - \left(\mathcal{G}_1 + A\frac{\partial\mathcal{G}_1}{\partial A} + 2B\frac{\partial\mathcal{G}_1}{\partial B}\right)\gamma_2 = 0\,, \tag{57}$$

$$\mathcal{G}_1\beta_2 + \frac{\partial\mathcal{G}_2}{\partial A}\varphi + 2A\mathcal{G}_2\beta_2 + B\left(2A\frac{\partial\mathcal{G}_2}{\partial B} - \frac{\partial\mathcal{G}_2}{\partial A}\right)\beta_2 + \left(2\mathcal{G}_2 + A\frac{\partial\mathcal{G}_2}{\partial A} + 2B\frac{\partial\mathcal{G}_2}{\partial B}\right)\gamma_1 = \mathcal{G}_2\,, \tag{58}$$

$$\mathcal{G}_1\beta_1 + \left(2\mathcal{G}_2 + 2B\frac{\partial\mathcal{G}_2}{\partial B}\right)\varphi + 2A\mathcal{G}_2\beta_1 + \left(2A\frac{\partial\mathcal{G}_2}{\partial B} - \frac{\partial\mathcal{G}_2}{\partial A}\right)B\beta_1 - \left(2\mathcal{G}_2 + \frac{\partial\mathcal{G}_2}{\partial A}A + 2B\frac{\partial\mathcal{G}_2}{\partial B}\right)B\gamma_2 = \mathcal{G}_1\,. \tag{59}$$

Equations (55) and (20) imply that the inverse shift operator has the simple realization

$$Z^{-1} = \mathcal{G}_1\varphi\,. \tag{60}$$

The system of PDEs for the unknown functions φ, β_i, γ_i, $\mathcal{F}_i$ and $\mathcal{G}_i$ is too difficult to solve in full generality. We will reduce the system to a manageable form by considering special choices for the functions β_1 and β_2: $\beta_1=\beta_2=0$, and $\beta_1=1$ and $\beta_2=0$. In each case, we obtain an infinite family of realizations parametrized by the function φ. We call these realizations type I and type II, respectively. In realization, (47)–(51) imply

$$\hat{x}_\mu = x_\mu\varphi + \mathrm{i}(x\partial)\left(a_\mu\gamma_1 + \mathrm{i}a^2\partial_\mu\gamma_2\right)\,, \tag{61}$$

where

$$\gamma_1 = \frac{\left(1+\frac{\partial\varphi}{\partial A}\right)\varphi}{\varphi - \left(A\frac{\partial\varphi}{\partial A} + 2B\frac{\partial\varphi}{\partial B}\right)}\,, \tag{62}$$

$$\gamma_2 = -\frac{2\frac{\partial\varphi}{\partial B}\varphi}{\varphi - \left(A\frac{\partial\varphi}{\partial A} + 2B\frac{\partial\varphi}{\partial B}\right)}\,. \tag{63}$$

Furthermore, (56)–(59) yield

$$\mathcal{G}_1 = \frac{1}{\varphi + A}, \quad \mathcal{G}_2 = \frac{1}{2\varphi(\varphi + A)}\,. \tag{64}$$

We note that, in view of (60), the shift operator is given by

$$Z = 1 + \frac{A}{\varphi}\,. \tag{65}$$

We are now able to find realizations of the rotation generators $M_{\mu\nu}$. Inserting the realizations for $\hat{x}_\mu$, D_μ and Z into (23) and comparing the obtained expression with (52), we find

$$\mathcal{F}_1 = 1\,,$$
$$\mathcal{F}_2 = \frac{\gamma_1}{\varphi} + \frac{B\gamma_2}{2\varphi^2} = \frac{\left(1+\frac{\partial\varphi}{\partial A}\right)\varphi - B\frac{\partial\varphi}{\partial B}}{\varphi^2 - \left(A\frac{\partial\varphi}{\partial A} + 2B\frac{\partial\varphi}{\partial B}\right)\varphi}\,,$$
$$\mathcal{F}_3 = \frac{1}{2\varphi}\,. \tag{66}$$

Note that $\mathcal{F}_1 = 1$ is consistent with the expression earlier obtained in (53). Next, we find a realization of the Laplace operator $\square$ that is uniquely determined by (13). Using the realizations for Z and D_μ in (64) and (65), we obtain $\sqrt{1-a^2D^2} = 1 - a^2\partial^2\mathcal{G}_2$. Therefore, (13) yields

$$\square = \partial^2\mathcal{H}\,, \quad \text{where} \quad \mathcal{H} = \frac{1}{\varphi(\varphi + A)}\,. \tag{67}$$

Note that, since $\varphi(0,0)=1$ and $A\to 0$ as $a\to 0$, in the limit we have $\square\to\partial^2$ as $a\to 0$. For realization II we have $\beta_1=1$ and $\beta_2=0$. Repeating the above calculations in the second realization we find

$$\hat{x}_\mu = x_\mu \varphi + \mathrm{i}(ax)\partial_\mu + \mathrm{i}(a\partial)\left(a_\mu \gamma_1 + \mathrm{i}a^2 \partial_\mu \gamma_2\right) , \quad (68)$$

where

$$\gamma_1 = \frac{\left(1+\frac{\partial\varphi}{\partial A}\right)\varphi}{\varphi - \left(A\frac{\partial\varphi}{\partial A} + 2B\frac{\partial\varphi}{\partial B}\right)} , \quad (69)$$

$$\gamma_2 = \frac{\frac{\partial\varphi}{\partial A} - 2(\varphi+A)\frac{\partial\varphi}{\partial B}}{\varphi - \left(A\frac{\partial\varphi}{\partial A} + 2B\frac{\partial\varphi}{\partial B}\right)} . \quad (70)$$

The functions $\mathcal{G}_1$ and $\mathcal{G}_2$ are given by

$$\mathcal{G}_1 = \frac{1}{\sqrt{(\varphi+A)^2+B}} , \quad \mathcal{G}_2 = 0 , \quad (71)$$

and the shift operator yields

$$Z = \sqrt{\left(1+\frac{A}{\varphi}\right)^2 + \frac{B}{\varphi^2}} . \quad (72)$$

Similarly, for $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$ we obtain

$$\mathcal{F}_1 = 1 ,$$
$$\mathcal{F}_2 = \frac{\gamma_1}{\varphi} = \frac{1+\frac{\partial\varphi}{\partial A}}{\varphi - \left(A\frac{\partial\varphi}{\partial A} + 2B\frac{\partial\varphi}{\partial B}\right)} ,$$
$$\mathcal{F}_3 = 0. \quad (73)$$

Note that $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$ in all realizations are consistent with (23). Furthermore, for the Laplace operator we have $\Box = \partial^2 \mathcal{H}$ where

$$\mathcal{H} = \frac{2}{B}\left(1-(\varphi+A)\mathcal{G}_1\right) = \frac{2}{B}\left(1 - \frac{\varphi+A}{\sqrt{(\varphi+A)^2+B}}\right) . \quad (74)$$

3.1 Special realizations

Of particular interest are some realizations obtained for a special choice of the parameter function φ. In the realization of type I we consider the *left*, *right* and *symmetric* realizations corresponding to $\varphi_\mathrm{L} = 1-A$, $\varphi_\mathrm{R} = 1$ and $\varphi_\mathrm{S} = A/(\exp(A)-1)$, respectively. One can show that in the left realization the derivative operator ∂_μ becomes the left deformation derivative ∂^L_μ. Similarly, in the right realization the derivative operator ∂_μ becomes the right deformation derivative ∂^R_μ. The symmetric realization is related to Weyl's symmetric ordering of the monomials in $\hat{x}_\mu$. In the realization of type II we consider the *natural* realization corresponding to $\varphi_\mathrm{N} = -A+\sqrt{1-B}$. In this realization the Dirac derivative is simply given by $D_\mu = \partial_\mu$.

For the left realization we have $\beta_1 = \beta_2 = 0$ and $\varphi_\mathrm{L} = 1-A$. Inserting $\varphi_\mathrm{L} = 1-A$ into (61)–(67), we find

$$\hat{x}_\mu = x_\mu(1-A) , \quad (75)$$
$$M_{\mu\nu} = x_\mu\partial_\nu - x_\nu\partial_\mu + \frac{1}{2}\mathrm{i}(x_\mu a_\nu - x_\nu a_\mu)\frac{1}{1-A}\partial^2 , \quad (76)$$
$$D_\mu = \partial_\mu + \frac{\mathrm{i}a_\mu}{2}\Box , \quad (77)$$
$$Z = \frac{1}{1-A} , \quad (78)$$
$$\Box = \frac{1}{1-A}\partial^2 . \quad (79)$$

It follows from (75) and (78) that

$$[\partial_\mu, \hat{x}_\nu] = \delta_{\mu\nu} Z^{-1} . \quad (80)$$

Thus, in view of (30), we see that ∂_μ is the left deformation derivative ∂^L_μ. For the right realization we have $\beta_1 = \beta_2 = 0$ and $\varphi_\mathrm{R} = 1$. Repeating the calculations with $\varphi_\mathrm{R} = 1$, we obtain

$$\hat{x}_\mu = x_\mu + \mathrm{i}a_\mu(x\partial) , \quad (81)$$
$$M_{\mu\nu} = x_\mu\partial_\nu - x_\nu\partial_\mu + \mathrm{i}(x\partial)(a_\mu\partial_\nu - a_\nu\partial_\mu) + \frac{\mathrm{i}}{2}(x_\mu a_\nu - a_\nu x_\mu)\partial^2 , \quad (82)$$
$$D_\mu = \frac{1}{1+A}\partial_\mu + \frac{\mathrm{i}a_\mu}{2}\Box , \quad (83)$$
$$Z = 1+A , \quad (84)$$
$$\Box = \frac{1}{1+A}\partial^2 . \quad (85)$$

In this case,

$$[\partial_\mu, \hat{x}_\nu] = \delta_{\mu\nu} + \mathrm{i}a_\nu\partial_\mu ; \quad (86)$$

hence, a comparison with (32) shows that ∂_μ is the right deformation derivative ∂^R_μ. For the symmetric realization we have $\beta_1 = \beta_2 = 0$ and $\varphi_\mathrm{S} = A/(\exp(A)-1)$.

This realization corresponds to the symmetric Weyl ordering [28]. It also follows from the universal formula for a general Lie algebra [59], after inserting the structure constants from (2). We have

$$\hat{x}_\mu = x_\mu \frac{A}{\mathrm{e}^A - 1} + \mathrm{i}a_\mu(x\partial)\frac{\mathrm{e}^A - 1 - A}{(\mathrm{e}^A-1)A} , \quad (87)$$
$$M_{\mu\nu} = x_\mu\partial_\nu - x_\nu\partial_\mu + \mathrm{i}(x\partial)(a_\mu\partial_\nu - a_\nu\partial_\mu)\frac{\mathrm{e}^A-1-A}{A^2} , \quad (88)$$
$$D_\mu = \frac{\mathrm{e}^A-1}{A\mathrm{e}^A}\partial_\mu + \frac{\mathrm{i}a_\mu}{2}\Box , \quad (89)$$
$$Z = \mathrm{e}^A , \quad (90)$$
$$\Box = \frac{(\mathrm{e}^A-1)^2}{A^2\mathrm{e}^A}\partial^2 \quad (91)$$

and

$$[\partial_\mu, \hat{x}_\nu] = \delta_{\mu\nu}\varphi_\mathrm{S} + \mathrm{i}a_\nu\partial_\mu\frac{1-\varphi_\mathrm{S}}{A} .$$

For the natural realization we have $\beta_1 = 1$, $\beta_2 = 0$, and $\varphi_{\mathrm{N}} = -A+\sqrt{1-B}$. The natural realization is a special case of the realization of type II in which the Dirac derivative $D_\mu = \partial_\mu \mathcal{G}_1 + \mathrm{i}a_\mu \partial^2 \mathcal{G}_2$ simplifies to $D_\mu = \partial_\mu$. In view of (71), this holds when $\varphi_{\mathrm{N}} = -A+\sqrt{1-B}$. Then we have

$$\hat{x}_\mu = x_\mu \left(-A+\sqrt{1-B}\right) + \mathrm{i}(ax)\partial_\mu \,, \tag{92}$$

$$M_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu \,, \tag{93}$$

$$D_\mu = \partial_\mu \,, \tag{94}$$

$$Z = \frac{1}{-A+\sqrt{1-B}} \,, \tag{95}$$

$$\square = \frac{2}{B}\left(1-\sqrt{1-B}\right)\partial^2 \,. \tag{96}$$

In the natural realization it is easily shown that the rotation generators $M_{\mu\nu}$ are given by (23). Indeed, (92) and (94) imply that

$$(\hat{x}_\mu D_\nu - \hat{x}_\nu D_\mu) Z = (x_\mu \partial_\nu - x_\nu \partial_\mu)\varphi Z \,. \tag{97}$$

However, $\varphi Z = 1$; thus, (93) and (97) yield

$$M_{\mu\nu} = (\hat{x}_\mu D_\nu - \hat{x}_\nu D_\mu) Z \,. \tag{98}$$

3.2 Hermiticity

All relations of the type $[\hat{x}_\mu, \hat{x}_\nu]$, $[M_{\mu\nu}, M_{\lambda\rho}]$, $[M_{\mu\nu}, \hat{x}_\lambda]$, $[D_\mu, D_\nu]$, $[M_{\mu\nu}, D_\lambda]$ and $[D_\mu, \hat{x}_\nu]$, i.e. (1), (4), (6), (7), (8) and (9), are invariant under the formal antilinear involution:

$$\hat{x}_\mu^\dagger = \hat{x}_\mu \,, \quad D_\mu^\dagger = -D_\mu \,, \quad M_{\mu\nu}^\dagger = -M_{\mu\nu} \,, \qquad c^\dagger = \bar{c} \,, \quad c \in \mathbb{C} \,. \tag{99}$$

The order of elements in the product is inverted under the involution. The commutative coordinates x_μ and their derivatives ∂_μ also satisfy the involution property: $x_\mu^\dagger = x_\mu$ and $\partial_\mu^\dagger = -\partial_\mu$. Then the NC coordinates $\hat{x}_\mu$ are represented by hermitian operators. However, (45) is generally *not* hermitian. The hermitian representations are simply obtained by the following expression [28]:

$$\hat{x}_\mu^h = \frac{1}{2}\left(x_\alpha \Phi_{\alpha\mu} + (\Phi^\dagger)_{\mu\alpha} x_\alpha\right) . \tag{100}$$

However, the physical results do not depend on the choice of representation as long as there exists a smooth limit $\hat{x}_\mu \to x_\mu$ as $a \to 0$. Therefore, we restrict ourselves to nonhermitian realizations only.

4 Star product

Recall that in Sect. 3 a general Ansatz for the NC coordinates was introduced,

$$\hat{x}_\mu = x_\alpha \Phi_{\alpha\mu}(A,B) \,, \quad \Phi_{\alpha\mu}(0,0) = \delta_{\alpha\mu} \,. \tag{101}$$

Let us define the vacuum state by $|0\rangle = 1$ and $\partial_\mu|0\rangle = 0$ and fix the normalization condition by $\hat{x}_\mu|0\rangle = x_\mu$. For a given realization $\Phi_{\mu\nu}$, there is a unique map sending monomials in the NC coordinates $\hat{x}_\mu$ into polynomials of the commutative coordinates x_μ. This map is given by

$$\prod_{i=1}^{k} \hat{x}_{\mu_i}|0\rangle = P_k(x) \,, \tag{102}$$

where P_k is a polynomial of degree k. We also have the dual relation

$$\prod_{i=1}^{k} x_{\mu_i} = \hat{P}_k(\hat{x})|0\rangle \,, \tag{103}$$

where $\hat{P}_k$ is also a polynomial of degree k in $\hat{x}$. For example, in the left realization we have

$$\prod_{i=1}^{k} \hat{x}_{\mu_i}|0\rangle = x_{\mu_1}(x_{\mu_2} - \mathrm{i}a_{\mu_2})(x_{\mu_3} - \mathrm{i}2a_{\mu_3})\dots \times (x_{\mu_k} - \mathrm{i}(k-1)a_{\mu_k}) \,, \tag{104}$$

together with the dual relation

$$\prod_{i=1}^{k} x_{\mu_i} = \hat{x}_{\mu_1} Z \hat{x}_{\mu_2} \dots Z \hat{x}_{\mu_k}|0\rangle \,. \tag{105}$$

Similarly, in the right realization we find

$$\prod_{i=1}^{k} \hat{x}_{\mu_i}|0\rangle = (x_{\mu_1} + \mathrm{i}(k-1)a_{\mu_1})(x_{\mu_2} + \mathrm{i}(k-2)a_{\mu_2})\dots \times (x_{\mu_{k-1}} + \mathrm{i}a_{\mu_{k-1}})x_{\mu_k} \,, \tag{106}$$

$$\prod_{i=1}^{k} x_{\mu_i} = Z^{-(k-1)} \hat{x}_{\mu_1} Z \hat{x}_{\mu_2} \dots Z \hat{x}_{\mu_k}|0\rangle \,. \tag{107}$$

One can obtain similar expressions for the symmetric realization [28]. It is interesting to note that in the realization of type I when $\varphi = \varphi(A)$ the following relation holds:

$$\mathrm{e}^{\mathrm{i}k\hat{x}}|0\rangle = \exp\left(\mathrm{i}\frac{\varphi(-ak)}{\varphi_{\mathrm{S}}(-ak)} kx\right) , \tag{108}$$

where $k \in \mathbb{R}^n$ and $\varphi_{\mathrm{S}}(A) = A/(\exp(A)-1)$. In particular, in the symmetric realization when $\varphi = \varphi_{\mathrm{S}}$, we have

$$\mathrm{e}^{\mathrm{i}k\hat{x}}|0\rangle = \mathrm{e}^{\mathrm{i}kx} \,. \tag{109}$$

Similarly, in the natural realization one can show that

$$\mathrm{e}^{\mathrm{i}k\hat{x}}|0\rangle = \mathrm{e}^{\mathrm{i}P_{\mathrm{N}}(k)x} \,, \tag{110}$$

where

$$P_{\mathrm{N}}(k)_\mu = \frac{1}{\varphi_{\mathrm{S}}(ak)}\left(k_\mu - \frac{k^2}{2\varphi_{\mathrm{S}}(-ak)} a_\mu\right) . \tag{111}$$

Equation (102) defines an isomorphism of the universal enveloping algebras generated by $\hat{x}_1, \hat{x}_2, \dots, \hat{x}_n$ and

$x_1, x_2, \ldots, x_n$, respectively. This can be extended to a formal power series by

$$\hat{f}(\hat{x})|0\rangle = f(x)\,, \tag{112}$$

where the function f depends on the realization $\Phi_{\mu\nu}$. The Leibniz rule and the related coproduct $\triangle\partial_\mu$ follow uniquely from the commutator relation

$$[\partial_\mu, \hat{x}_\nu] = \Phi_{\mu\nu}(A, B) \tag{113}$$

and the conditions $\partial_\mu|0\rangle = 0$ and $\hat{x}_\mu|0\rangle = x_\mu$.

The star product of the two functions $f(x)$ and $g(x)$ is defined by

$$(f \star g)(x) = \hat{f}(\hat{x})\hat{g}(\hat{x})|0\rangle\,. \tag{114}$$

We emphasize that the star product depends on the realization $\Phi_{\mu\nu}$. The result relating the star product and the coproduct obtained for the noncovariant realizations [28] can be extended to covariant realizations of κ-deformed space, (1), as follows:

$$(f \star g)(u) = m\left(\mathrm{e}^{u_\alpha(\triangle - \triangle_0)\partial_\alpha} f(x) \otimes g(y)\right)\Big|_{\substack{x=u\\y=u}}, \tag{115}$$

where m is the multiplication map in the Hopf algebra, and $\triangle_0\partial_\mu = \partial_\mu \otimes \mathbf{1} + \mathbf{1} \otimes \partial_\mu$ is the undeformed coproduct.

4.1 Star product for the realization of type I

In this subsection we discuss the star product in the realization of type I assuming that the parameter function φ depends only on the variable $A = \mathrm{i}a\partial$. In view of (61), we have

$$\hat{x}_\mu = x_\mu\varphi(A) + \mathrm{i}(x\partial)a_\mu\gamma(A)\,, \tag{116}$$

where

$$\gamma(A) = \frac{1+\varphi'(A)}{1 - A\frac{\varphi'(A)}{\varphi(A)}}\,, \quad \varphi(0) = 1\,. \tag{117}$$

In order to find the coproduct $\triangle\partial_\mu$, we note that in this realization $\partial_\mu = \varphi(A)\partial_\mu^{\mathrm{R}}$. The coproduct for $\partial_\mu^{\mathrm{R}}$ is given by (34); hence

$$\begin{aligned}\triangle\partial_\mu &= \triangle\varphi(A)\triangle\partial_\mu^{\mathrm{R}} \\ &= \triangle\varphi(A)\left[\frac{1}{\varphi(A)}\partial_\mu \otimes \mathbf{1} + Z \otimes \frac{1}{\varphi(A)}\partial_\mu\right].\end{aligned} \tag{118}$$

Inverting the expression for the shift operator $Z = 1 + A/\varphi(A)$, we find

$$A = (Z-1) + \varphi'(0)(Z-1)^2 + \cdots\,, \tag{119}$$

which, together with $\triangle Z = Z \otimes Z$, allows us to calculate the coproduct $\triangle\varphi(A)$. Then, to second order in the parameter a, (118) leads to

$$\begin{aligned}\triangle\partial_\mu = \partial_\mu^x &\Big[1 + \varphi'(0)A_y + \left(\varphi''(0) + (\varphi'(0))^2 + \varphi'(0)\right)A_xA_y \\ &\quad + \frac{1}{2}\varphi''(0)A_y^2\Big] \\ &+ \partial_\mu^y\Big[1 + (1+\varphi'(0))A_x \\ &\quad + \left(\varphi''(0) + (\varphi'(0))^2 + \varphi'(0)\right)A_xA_y \\ &\quad + \frac{1}{2}\varphi''(0)A_x^2\Big]\,,\end{aligned} \tag{120}$$

where $A_x = \mathrm{i}a\partial^x$ and $A_y = \mathrm{i}a\partial^y$. Consequently, the star product from (115) is given by

$$\begin{aligned}(f \star g)(u) &= \Big\{1 + u\partial^x\Big[\varphi'(0)A_y + \left(\varphi''(0) + (\varphi'(0))^2 + \varphi'(0)\right)A_xA_y \\ &\quad + \frac{1}{2}\varphi''(0)A_y^2\Big] \\ &+ u\partial_y\Big[(1+\varphi'(0))A_x + \left(\varphi''(0) + (\varphi'(0))^2 + \varphi'(0)\right)A_xA_y \\ &\quad + \frac{1}{2}\varphi''(0)A_x\Big] \\ &+ \frac{1}{2}\left[\varphi'(0)u\partial^xA_y + (1+\varphi'(0))u\partial^yA_x\right]^2\Big\} f(x)g(y)\Big|_{\substack{x=u\\y=u}}.\end{aligned} \tag{121}$$

One can show that the dual relation holds

$$(f \star g)_{\varphi(A)} = (g \star f)_{\varphi(-A)-A}\,, \tag{122}$$

where the star products correspond to the functions $\varphi(A)$ and $\varphi(-A) - A$, respectively.

4.2 Star product for special realizations

In this subsection we give the star products in closed form for the left, right and symmetric realizations, as well as the star product to second order in a in the natural realization. For the left realization we have $\varphi_{\mathrm{L}} = 1 - A$; now

$$(f \star g)_{\varphi_{\mathrm{L}}}(u) = \mathrm{e}^{-u_\alpha\partial_\alpha^x A_y} f(x)g(y)\Big|_{\substack{x=u\\y=u}}. \tag{123}$$

For the right realization we have $\varphi_{\mathrm{R}} = 1$; now

$$(f \star g)_{\varphi_{\mathrm{R}}}(u) = \mathrm{e}^{u_\alpha\partial_\alpha^y A_x} f(x)g(y)\Big|_{\substack{x=u\\y=u}}. \tag{124}$$

The "left" and "right" star products satisfy the symmetry relation

$$(f \star g)_{\varphi_{\mathrm{L}}} = (g \star f)_{\varphi_{\mathrm{R}}}\,. \tag{125}$$

For the symmetric realization we have $\varphi_{\rm S}=A/(\exp(A)-1)$; now

$$(f\star g)_{\varphi_{\rm S}}(u)=\mathrm{e}^{u_\alpha(\triangle-\triangle_0)\partial_\alpha}f(x)g(y)\Big|_{\substack{x=u\\y=u}}, \qquad (126)$$

where

$$\triangle_0\partial_\mu=\partial^x_\mu+\partial^y_\mu, \qquad (127)$$

$$\triangle\partial_\mu=\partial^x_\mu\frac{\varphi_{\rm S}(A_x+A_y)}{\varphi_{\rm S}(A_x)}+\partial^y_\mu\frac{\varphi_{\rm S}(-A_x-A_y)}{\varphi_{\rm S}(-A_y)}. \qquad (128)$$

In this case, we have

$$(f\star g)_{\varphi_{\rm S}(A)}=(g\star f)_{\varphi_{\rm S}(-A)}. \qquad (129)$$

The symmetric realization $\varphi=\varphi_{\rm S}$ corresponds to the symmetric Weyl ordering [28]. Our closed form results, (126) and (128), are in agreement with the general series expansion for the star product of the Lie algebra type NC space [60]. For the natural realization we have $\varphi_{\rm N}=-A+\sqrt{1-B}$. In this realization $\partial_\mu=D_\mu$; hence the coproduct $\triangle\partial_\mu$ is given by (27). One can show that to second order in the parameter a the star product yields

$$\begin{aligned}(f\star g)_{\varphi_{\rm N}}(u)\\ =f(u)g(u)+\Bigg\{u_\mu\Bigg[\left(-\partial^x_\mu-\frac{\mathrm{i}a_\mu}{2a^2}a^2_\alpha(\partial^x_\alpha)^2\right)A_y\\ -\frac{1}{2}\partial^x_\mu a^2_\alpha(\partial^y_\alpha)^2+\mathrm{i}a_\mu(1+A_x)\partial^x\partial^y\Bigg]\\ +\frac{1}{2}u_\mu u_\nu\left[\partial^x_\mu\partial^y_\nu A^2_y-2\mathrm{i}a_\mu\partial^x_\nu\partial^x\partial^y A_y-a_\mu a_\nu(\partial^x\partial^y)^2\right]\Bigg\}\\ \times f(x)g(y)\Big|_{\substack{x=u\\y=u}},\end{aligned} \qquad (130)$$

where $\partial^x\partial^y=\partial^x_\alpha\partial^y_\alpha$.

4.3 Equivalent star products

So far, we have considered the star product for some specific realizations of type I and II. We point out, however, that infinitely many realizations of the star product can be constructed by similarity transformations of the variables x_μ and ∂_μ. In the following we consider the star products obtained by similarity transformations starting from the right realization.

Recall that in the right realization we have

$$\hat{x}_\mu=x^{\rm R}_\mu+\mathrm{i}a_\mu(x^{\rm R}\partial^{\rm R}), \qquad (131)$$

and the star product is given by (124). (From now on the variables x_μ and ∂_μ used in the right realization will be denoted by $x^{\rm R}_\mu$ and $\partial^{\rm R}_\mu$, respectively.) The similarity transformation is defined by

$$x_\mu=S^{-1}x^{\rm R}_\mu S, \qquad (132)$$

$$\partial_\mu=S^{-1}\partial^{\rm R}_\mu S. \qquad (133)$$

Clearly, the new variables x_μ and ∂_μ also generate the Heisenberg algebra. We define the vacuum condition on S by $S|0\rangle=|0\rangle$. In view of (124), the star product induced by the similarity transformation S is given by

$$\begin{aligned}(f\underset{S}{\star}g)(u)&=S\left(S^{-1}f\star S^{-1}g\right)_{\varphi_{\rm R}}(u)\\ &=S\mathrm{e}^{u_\alpha\partial^y_\alpha A_x}(S^{-1}f)(x)(S^{-1}g)(y)\Big|_{\substack{x=u\\y=u}}.\end{aligned} \qquad (134)$$

Two star products are said to be equivalent if they are related by a similarity transformation. For example, if $S=\mathrm{e}^{-x\partial A_x}$, then

$$(f\star g)_{\varphi_{\rm L}}(u)=S\left(S^{-1}f\star S^{-1}g\right)_{\varphi_{\rm R}}(u); \qquad (135)$$

hence, the star products for the left and right realization are equivalent. We will show that all realizations of type I with $\varphi=\varphi(A)$ lie in the orbits of the action of similarity transformations of the right realization. Hence, any two star products in realizations of this type are equivalent.

Consider a realization of type I,

$$\hat{x}_\mu=x_\mu\varphi(A)+\mathrm{i}a_\mu(x\partial)\frac{1+\varphi'(A)}{1-A\frac{\varphi'(A)}{\varphi(A)}}, \quad \varphi(0)=1. \qquad (136)$$

The transformation $(x_\mu,\partial_\mu)\mapsto(x^{\rm R}_\mu,\partial^{\rm R}_\mu)$, which maps the realization (131) into (136), is given by

$$x^{\rm R}_\mu=x_\mu\varphi(A)+\mathrm{i}a_\mu(x\partial)\frac{\varphi'(A)}{1-A\frac{\varphi'(A)}{\varphi(A)}}, \qquad (137)$$

$$\partial^{\rm R}_\mu=\partial_\mu\frac{1}{\varphi(A)}. \qquad (138)$$

We show that there exists a similarity operator of the form $S=\exp(U)$,

$$U=(x\partial)\sum_{k=1}^{\infty}c_kA^k, \qquad (139)$$

such that (137) and (138) are given by $x^{\rm R}_\mu=Sx_\mu S^{-1}$ and $\partial^{\rm R}_\mu=S\partial_\mu S^{-1}$, respectively. Then (138) yields

$$\exp(\mathrm{ad}(U))\partial_\mu=\partial_\mu\frac{1}{\varphi(A)}. \qquad (140)$$

By expanding both sides of (140) into a power series in A, one can show that the coefficients c_k are uniquely determined by the function $\varphi(A)$. Expanding the right-hand side of (140) leads to

$$\partial_\mu\frac{1}{\varphi(A)}=\partial_\mu\left(1+\sum_{p=1}^{\infty}\frac{1}{p!}\varepsilon_pA^p\right), \qquad (141)$$

where the coefficients ε_p can be found recursively from

$$\varepsilon_1 = -\varphi'(0)\,, \tag{142}$$

$$\varepsilon_p = -\varphi^{(p)}(0) - \sum_{k=1}^{p-1} \binom{p}{k} \varphi^{(k)}(0)\varepsilon_{p-k}\,, \quad p \geq 2\,. \tag{143}$$

Similarly, the expansion of the left-hand side of (140) gives

$$\exp(\mathrm{ad}(U))\partial_\mu = \partial_\mu \left(1 + \sum_{p=1}^{\infty} \gamma_p A^p\right)\,, \tag{144}$$

where the coefficients γ_p are found in terms of c_k, as follows:

$$\gamma_1 = -c_1\,, \tag{145}$$

$$\gamma_p = -c_p + \sum_{n=2}^{p} \frac{(-1)^n}{n!} \beta_p^{(n)}\,, \quad p \geq 2\,. \tag{146}$$

Here,

$$\beta_p^{(n)} = \sum_{|k|=p} \Psi(k) \prod_{i=1}^{n} c_{k_i}\,, \tag{147}$$

where $k = (k_1, k_2, \dots, k_n)$, $k_i \in \mathbb{N}$, is a multi-index with length $|k| = \sum_{i=1}^{n} k_i$, the function $\Psi(k)$ is defined by

$$\Psi(k) = (1 + k_n)(1 + k_n + k_{n-1}) \dots \left(1 + \sum_{k=2}^{n} k_i\right)\,, \tag{148}$$

and the summation is taken over all multi-indices such that $|k| = p$. It follows from (141) and (144) that

$$c_1 = -\varepsilon_1\,, \tag{149}$$

$$c_p = \sum_{n=2}^{p} \frac{(-1)^n}{n!} \beta_p^{(n)} - \frac{1}{p!}\varepsilon_p\,, \quad p \geq 2\,. \tag{150}$$

Note that the coefficient c_p is uniquely determined by $c_1, c_2, \dots, c_{p-1}$. Hence, (149) and (150), together with (142) and (143), provide recursion relations for c_p in terms of $\varphi^{(k)}(0)$; thus S is uniquely determined by $\varphi(A)$. For example, the first three coefficients are given by $c_1 = \varphi'(0)$, $c_2 = \frac{1}{2}\varphi''(0)$, $c_3 = \frac{1}{6}\varphi'''(0) + \frac{1}{4}\varphi'(0)\varphi''(0)$. Equations (134) and (139), together with c_1 and c_2, reproduce the star product given by (121). This represents an important consistency check of our approach.

Note that our covariant realizations for $\hat{x}_\mu$, (47), and for D_μ, (54), follow from $S = \mathrm{e}^U$, where

$$U = (x\partial)\Phi_1(A,B) + (xa)\partial^2\Phi_2(A,B)\,,$$

with the boundary conditions of $\Phi_1(0,0) = 0$ and of $\Phi_2(0,0)$ being finite.

4.4 Scalar fields and invariants on kappa-deformed space

All covariant realizations are physically equivalent. Here we consider the natural realization x^{N} and ∂^{N} defined by $D_\mu = \partial_\mu^{\mathrm{N}}$ and $M_{\mu\nu} = x_\mu^{\mathrm{N}}\partial_\nu^{\mathrm{N}} - x_\nu^{\mathrm{N}}\partial_\mu^{\mathrm{N}}$ (c.f. Sects. 3.1 and 4.2). The realization of the NC coordinates is given by (92),

$$\hat{x}_\mu = x_\mu^{\mathrm{N}} Z^{-1} + \mathrm{i}(ax^{\mathrm{N}})\partial_\mu^{\mathrm{N}}\,. \tag{151}$$

Let us consider a scalar field $\hat{\Phi}(\hat{x})$ in the NC coordinates satisfying $[M_{\mu\nu}, \hat{\Phi}(\hat{x})] = 0$ for all μ, ν. We define the scalar field $\Phi(x^{\mathrm{N}})$ in the undeformed space by

$$\hat{\Phi}\left(\hat{x}(x^{\mathrm{N}})\right)|0\rangle = \Phi(x^{\mathrm{N}})\,. \tag{152}$$

The ordinary Fourier transform is defined by

$$\tilde{\Phi}(k) = \int \mathrm{d}^n x^{\mathrm{N}} \mathrm{e}^{-\mathrm{i}kx^{\mathrm{N}}} \Phi(x^{\mathrm{N}})\,, \quad k \in \mathbb{R}^n\,. \tag{153}$$

Then using the relation (110),

$$\mathrm{e}^{\mathrm{i}k\hat{x}}|0\rangle = \mathrm{e}^{\mathrm{i}P_{\mathrm{N}}(k)x^{\mathrm{N}}}\,, \tag{154}$$

where $P_{\mathrm{N}}(k)$ is given by (111), we find

$$\hat{\Phi}(\hat{x}) = \int \mathrm{d}^n k \tilde{\Phi}(k) \mathrm{e}^{\mathrm{i}P_{\mathrm{N}}^{-1}(k)\hat{x}}\,, \tag{155}$$

which holds in any realization of $\hat{x}$. Here P_{N}^{-1} denotes the inverse function of P_{N},

$$P_{\mathrm{N}}^{-1}(k)_\mu = \frac{\ln Z^{-1}(k)}{Z^{-1}(k) - 1}\left(k_\mu + \frac{a_\mu}{a^2}(\sqrt{1 + a^2k^2} - 1)\right)\,,$$

where

$$Z^{-1}(k) = \sqrt{1 + a^2k^2} + ak\,.$$

The above relation, (155), represents a construction of the $\mathrm{SO}_a(n)$ invariants $\hat{\Phi}(\hat{x})$ in terms of $\Phi(x^{\mathrm{N}})$ and $\tilde{\Phi}(k)$ by using the natural realization (92). Alternatively, from (92) we obtain the inverse mapping

$$x_\mu^{\mathrm{N}} = \left(\hat{x}_\mu - \mathrm{i}(a\hat{x}) \frac{\partial_\mu^{\mathrm{N}}}{\sqrt{1 - a^2(\partial_\mu^{\mathrm{N}})^2}}\right) Z\,. \tag{156}$$

Then we find

$$\Phi\left(x^{\mathrm{N}}(\hat{x})\right)|0\rangle = \hat{\Phi}(\hat{x})\,. \tag{157}$$

Both constructions are consistent, and they are equivalent. Furthermore, we define the invariant integration over the entire NC space using the natural realization, as follows:

$$\int \hat{\Phi}(\hat{x}) = \int \mathrm{d}^n x^{\mathrm{N}} \Phi(x^{\mathrm{N}})\,, \tag{158}$$

with the property

$$\int \hat{\Phi}_1(\hat{x})\hat{\Phi}_2(\hat{x}) = \int \mathrm{d}^n x^{\mathrm{N}} (\Phi_1 \star \Phi_2)_{\mathrm{N}}(x^{\mathrm{N}}) \,. \tag{159}$$

A generalized action of the scalar field $\Phi(\hat{x})$ on the NC κ-deformed space is simply the action of the ordinary scalar field in natural coordinates x^{N} on the undeformed space in which pointwise multiplication of fields is replaced by $\star$-mulitplication in the natural realization. Further investigation of this problem is in progress and will be published separately.

5 Conclusion

We have constructed covariant realizations of a general κ-deformed space in terms of commutative coordinates x_μ and their derivatives ∂_μ in the undeformed space. Our construction can also be applied to spaces with arbitrary signatures, especially to Minkowski type spaces.

Particularly, we have studied the κ-deformed space whose deformation is described by an arbitrary vector. The NC coordinates and rotation generators form an extended Lie algebra. The subalgebra of the rotation generators, $\mathrm{SO}_a(n)$, is undeformed. The Dirac derivatives mutually commute and are vector-like under the action of $\mathrm{SO}_a(n)$. By introducing the shift operator the deformed Heisenberg algebra is written in a very simple way. We have presented the Leibniz rule and coproduct for the rotation generators $M_{\mu\nu}$ and the Dirac derivatives D_μ in a covariant form.

We have found two types of covariant realizations, which are described by an arbitrary function $\varphi(A)$ with $\varphi(0) = 1$. We point out that all covariant realizations are equivalent and can be treated on an equal footing. We have constructed coproducts and star products for covariant realizations. There is an important relation between the coproduct and the star product in terms of an exponential map for a given realization. Specially, we have found a few realizations (covariant left, right, symmetric and natural) which have very simple properties. All realizations of type I are related by similarity transformations and the corresponding star products are equivalent. We have considered scalar fields in NC coordinates and demonstrated their simple properties using the natural realization. The constructions of invariants and invariant integration on NC spaces are also discussed.

Our approach may be useful in quantum gravity models, specially in $2+1$ dimensions. In this case, the corresponding Lie algebra is SU(2) or SU(1,1) [61–64]. It would be interesting to classify NC spaces with covariant realizations in which NC coordinates and rotation generators form an extended Lie algebra. For example, Snyder space is of this type, and covariant realizations in terms of undeformed space exist [65].

Acknowledgements. One of us (S.M.) thanks D. Svrtan and Z. Škoda for useful discussions. This work is supported by the Ministry of Science and Technology of the Republic of Croatia under the contracts 098-0000000-2865 and 0023003.

References

1. S. Doplicher, K. Fredenhagen, J.E. Roberts, Phys. Lett. B **331**, 39 (1994)
2. S. Doplicher, K. Fredenhagen, J.E. Roberts, Commun. Math. Phys. **172**, 187 (1995) [hep-th/0303037]
3. N. Seiberg, E. Witten, JHEP **09**, 032 (1999) [hep-th/9908142]
4. J. de Boer, P.A. Grassi, P. van Nieuwenhuizen, Phys. Lett. B **574**, 98 (2003) [hep-th/0302078]
5. M.R. Douglas, N.A. Nekrasov, Rev. Mod. Phys. **73**, 977 (2001) [hep-th/0106048]
6. R.J. Szabo, Phys. Rep. **378**, 207 (2003) [hep-th/0109162]
7. R.J. Szabo, Class. Quantum Grav. **23**, 199 (2006) [hep-th/0606233]
8. P. Aschieri, B. Jurco, P. Schupp, J. Wess, Nucl. Phys. B **651**, 45 (2003) [hep-th/0205214]
9. P. Aschieri, C. Blohmann, M. Dimitrijević, F. Meyer, P. Schupp, J. Wess, Class. Quantum Grav. **22**, 3511 (2005) [hep-th/0504183]
10. J. Lukierski, A. Nowicki, H. Ruegg, V.N. Tolstoy, Phys. Lett. B **264**, 331 (1991)
11. J. Lukierski, A. Nowicki, H. Ruegg, Phys. Lett. B **293**, 344 (1992)
12. J. Lukierski, H. Ruegg, Phys. Lett. B **329**, 189 (1994) [hep-th/9310117]
13. S. Majid, H. Ruegg, Phys. Lett. B **334**, 348 (1994) [hep-th/9404107]
14. P. Kosiński, P. Maślanka, hep-th/9411033
15. A. Sitarz, Phys. Lett. B **349**, 42 (1995) [hep-th/9409014]
16. K. Kosiński, J. Lukierski, P. Maślanka, Phys. Rev. D **62**, 025 004 (2000) [hep-th/9902037]
17. K. Kosiński, J. Lukierski, P. Maślanka, Czech. J. Phys. **50**, 1283 (2000) [hep-th/0009120]
18. P. Kosiński, J. Lukierski, P. Maślanka, A. Sitarz, hep-th/0307038
19. G. Amelino-Camelia, Phys. Lett. **510**, 255 (2001) [hep-th/0012238]
20. G. Amelino-Camelia, Int. J. Mod. Phys. D **11**, 35 (2002) [gr-qc/0012051]
21. N.R. Bruno, G. Amelino-Camelia, J. Kowalski-Glikman, Phys. Lett. B **522**, 133 (2001) [hep-th/0107039]
22. J. Kowalski-Glikman, S. Nowak, Phys. Lett. B **539**, 126 (2002) [hep- th/0203040]
23. G. Amelino-Camelia, M. Arzano, Phys. Rev. D **65**, 084 044 (2002) [hep-th/0105120]
24. G. Amelino-Camelia, F. D'Andrea, G. Mandanici, JCAP **0309**, 006 (2003) [hep-th/0211022]
25. M. Dimitrijević, L. Jonke, L. Möller, E. Tsouchnika, J. Wess, M. Wohlgenannt, Eur. Phys. J. C **31**, 129 (2003) [hep-th/0307149]
26. M. Dimitrijević, F. Meyer, L. Möller, J. Wess, Eur. Phys. J. C **36**, 117 (2004) [hep-th/0310116]
27. M. Dimitrijević, L. Möller, E. Tsouchnika, J. Phys. A **37**, 9749 (2004) [hep-th/0404224]
28. S. Meljanac, M. Stojić, Eur. Phys. J. C **47**, 531 (2006) [hep-th/0605133]
29. L. Möller, JHEP **0512**, 029 (2005) [hep-th/0409128]
30. A. Agostini, G. Amelino-Camelia, M. Arzano, F. D'Andrea, hep-th/0407227
31. A. Agostini, G. Amelino-Camelia, M. Arzano, A. Marciano, R.A. Tacchi, hep-th/0607221

32. L. Freidel, J. Kowalski-Glikman, S. Nowak, hep-th/0612170
33. S. Ghosh, Phys. Lett. B **623**, 251 (2005) [hep-th/0506084]
34. S. Ghosh, P. Pal, Phys. Lett. B **618**, 243 (2005) [hep-th/0502192]
35. S. Ghosh, Phys. Rev. D **74**, 084 019 (2006) [hep-th/0608206]
36. H.-C. Kim, J.H. Yee, C. Rim, hep-th/0701054
37. D. Bonatsos, C. Daskaloyannis, Phys. Lett. B **307**, 100 (1993)
38. S. Meljanac, M. Mileković, S. Pallua, Phys. Lett. B **328**, 55 (1994) [hep-th/9404039]
39. S. Meljanac, M. Mileković, Int. J. Mod. Phys. A **11**, 1391 (1996)
40. S. Meljanac, A. Perica, J. Phys. A Math. Gen. **27**, 4737 (1994)
41. S. Meljanac, A. Perica, Mod. Phys. Lett. A **9**, 3293 (1994) [hep-th/9409180]
42. S. Meljanac, A. Perica, D. Svrtan, J. Phys. A **36**, 6337 (2003) [math-ph/0304038]
43. S. Meljanac, D. Svrtan, math-ph/0304040
44. S. Meljanac, D. Svrtan, Math. Commun. **1**, 1 (1996) [math-ph/0304039]
45. V. Bardek, S. Meljanac, Eur. Phys. J. C **17**, 539 (2000) [hep-th/0009099]
46. V. Bardek, L. Jonke, S. Meljanac, M. Mileković, Phys. Lett. B **531**, 311 (2002) [hep-th/0107053]
47. S. Meljanac, M. Mileković, M. Stojić, Eur. Phys. J. C **24**, 331 (2002) [math-ph/0201061]
48. L. Jonke, S. Meljanac, Phys. Lett. B **526**, 149 (2002) [hep-th/0106135]
49. V.P. Nair, A.P. Polychronakos, Phys. Lett. B **505**, 267 (2001) [hep-th/0011172]
50. L. Jonke, S. Meljanac, Eur. Phys. J. C **29**, 433 (2003) [hep-th/0210042]
51. I. Dadić, L. Jonke, S. Meljanac, Acta Phys. Slovaca **55**, 149 (2005) [hep-th/0301066]
52. A. Kempf, G. Mangano, R.B. Mann, Phys. Rev. D **52**, 1108 (1995) [hep-th/9412167]
53. L.N. Chang, D. Minic, N. Okamura, T. Takeuchi, Phys. Rev. D **65**, 125 027 (2002) [hep-th/0201017]
54. I. Dadić, L. Jonke, S. Meljanac, Phys. Rev. D **67**, 087 701 (2003) [hep-th/0210264]
55. J. Wess, Deformed coordinates spaces, Derivatives, Lecture given at the Balkan workshop BW2003, August 2003, Vrnjačka Banja, Serbia, [hep-th/0408080]
56. M. Chaichian, P.P. Kulish, K. Nishijima, A. Tureanu, Phys. Lett. B **604**, 98 (2004) [hep-th/0408069]
57. M. Chaichian, P. Presnajder, A. Tureanu, Phys. Rev. Lett. **94**, 151 602 (2005) [hep-th/0409096]
58. J. Lukierski, M. Woronowicz, Phys. Lett. B **633**, 116 (2006) [hep-th/0508083]
59. N. Durov, S. Meljanac, A. Samsarov, Z. Škoda, J. Algebra **309** (2007) [math.RT/0604096]
60. V. Kathotia, Int. J. Math. Math. Sci. **11**, 523 (2000) [math.qa/9811174]
61. L. Freidel, E.R. Livine, Class. Quantum Grav. **23**, 2021 (2006) [hep-th/0502106]
62. L. Freidel, E.R. Livine, Phys. Rev. Lett. **96**, 221 301 (2006) [hep-th/0512113]
63. S. Majid, J. Math. Phys. **46**, 103 520 (2005) [hep-th/0507271]
64. L. Freidel, S. Majid, hep/th-0601004
65. R. Banerjee, S. Kulkarni, S. Samatra, JHEP **0605**, 077 (2006) [hep-th/0602151]